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2000 J. Phys. A: Math. Gen. 33 8383

(http://iopscience.iop.org/0305-4470/33/47/305)

J. Phys. A: Math. Gen. **33** (2000) 8383–8388. Printed in the UK PII: S0305-4470(00)13275-5

Scaling and persistence in the two-dimensional Ising model

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Received 11 April 2000, in final form 26 September 2000

Abstract. The spatial distribution of persistent spins at zero temperature in the pure two-dimensional Ising model is investigated numerically. A persistence correlation length, $\xi(t) \sim t^Z$, is identified such that for length scales $r \ll \xi(t)$ the persistent spins form a fractal with dimension d_f; for length scales $r \gg \xi(t)$ the distribution of persistent spins is homogeneous. The zero-temperature persistence exponent, θ, is found to satisfy the scaling relation $\theta = Z(2 - d_f)$ with $\theta = 0.209 \pm 0.002$ of Jain (Jain S 1999 *Phys. Rev.* E **59** R2493), $Z = 1/2$ and $d_f \sim 1.58$.

The 'persistence' problem has attracted considerable interest in recent years [1–9]. In its most general form, it is concerned with the fraction of space which persists in its initial state up to some later time.

Hence, in the non-equilibrium dynamics of spin systems at zero temperature we are interested in the fraction of spins, $P(t)$, that persist in the same state as at $t = 0$ up to some later time t. For the pure ferromagnetic two-dimensional Ising model, $P(t)$ has been found to decay algebraically [1–4]

$$P(t) \sim t^{-\theta} \tag{1}$$

where $\theta = 0.209 \pm 0.002$ [5]. Similar algebraic decay has been found in numerous other systems displaying persistence [9]. Most of the recent theoretical effort has gone into obtaining the numerical value of θ for different models.

Very recently, Manoj and Ray [10] have studied the spatial correlation of persistent sites in the 1D $A + A \rightarrow 0$ model. They found that the set of persistent sites in their 1D model forms a fractal over sufficiently small length scales.

In this paper we present the results of an extensive numerical study of the spatial distribution of persistent spins in the pure 2D Ising model at zero temperature. As we will see, the 2D Ising model exhibits behaviour very similar to that found by Manoj and Ray [10] in their simple 1D model.

The Hamiltonian for our model is given by

$$H = -\sum_{\langle ij \rangle} S_i S_j \tag{2}$$

where $S_i = \pm 1$ are Ising spins situated on every site of a square lattice with periodic boundary conditions; the summation in equation (2) runs over all nearest-neighbour pairs only.

The data presented in this paper were obtained for a lattice with dimensions 1000×1000 ($=N$).

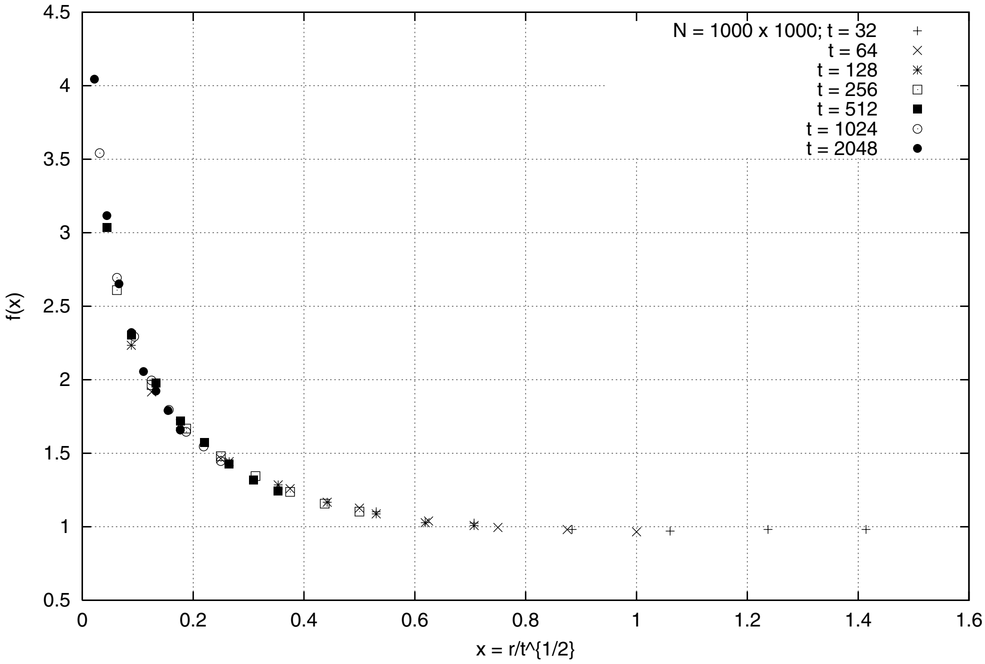

Figure 1. A plot of the scaling function $f(x)$ $(=C(r,t)/P(t))$ against x where $x = r/\sqrt{t}$ for t ranging over approximately three orders of magnitude.

Each simulation run begins at $t = 0$ with a random (± 1) starting configuration of the spins and then we update the lattice via single-spin-flip zero-temperature Glauber dynamics [5]. The rule we use is: always flip if the energy change is negative, never flip if the energy change is positive and flip at random if the energy change is zero.

For each spin S_i we define

$$n_i(t) = (S_i(t)S_i(0) + 1)/2. \tag{3}$$

Hence, $n_i(0) = 1$ for all i. Furthermore, $n_i(t)$ changes from 1 to 0 at the first flip of S_i and, in the simulations, *remains* at this value even if S_i undergoes subsequent flips. The spin S_i is said to be persistent if and only if $n_i(t) = 1$ for all $t \geqslant 0$.

The total number, $n(t)$, of spins which have never flipped until time t is then given by $n(t) = \sum_i n_i(t)$, and the persistence probability by [1]

$$P(t) = \sum_i \langle n_i(t)\rangle / N \tag{4}$$

where $\langle\cdots\rangle$ indicates averages over different initial conditions and histories. We averaged over at least 100 different initial conditions and histories for each run.

To investigate the spatial correlations in this model, we follow Manoj and Ray [10] and study the two-point correlator defined by

$$C(r,t) = \langle n_i(t)n_{i+r}(t)\rangle / \langle n_i(t)\rangle \tag{5}$$

where $\langle\cdots\rangle$ now also includes the average over the lattice shown explicitly in equation (4). $C(r,t)$ is simply the probability that a spin at site $i + r$ is persistent given that the one at site i is persistent, averaged over the entire lattice. As we are working with a square lattice, we

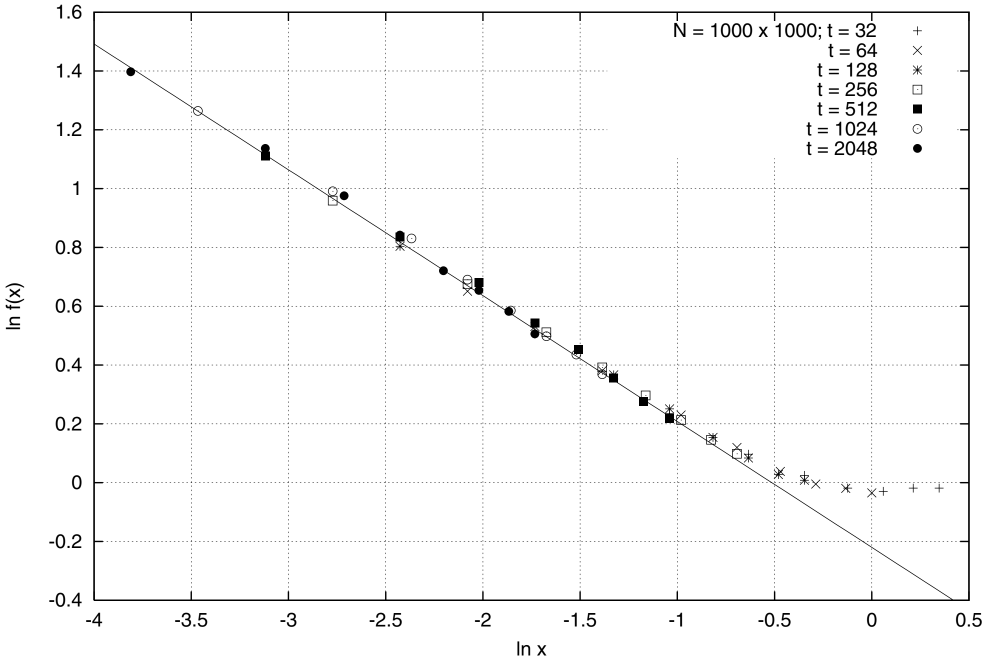

Figure 2. A re-plot of the data shown in figure 1 on a log–log scale. The straight line implies a value of $\alpha = 0.428 \pm 0.007$.

estimated n_{i+r} by considering spins displaced along the x and y directions from i. According to [10], the two-point correlator satisfies the following dynamic scaling relation:

$$C(r,t) = P(t) f(r/\xi(t)) \tag{6}$$

where $\xi(t)$ is the persistence correlation length and $f(x)$ is a scaling function such that

$$f(x) \sim \begin{cases} x^{-\alpha} & \text{for} \quad x \ll 1 \\ 1 & \text{for} \quad x \gg 1. \end{cases} \tag{7}$$

As $P(t) \sim t^{-\theta}$, for the two-point correlator to be independent of t for $r \ll \xi(t)$ we require $\xi^{-\alpha} \sim t^{-\theta}$. As a consequence, the expected behaviour of $C(r,t)$ in the two limits is given by

$$C(r,t) \sim \begin{cases} r^{-\alpha} & \text{for} \quad r \ll \xi(t) \\ t^{-\theta} & \text{for} \quad r \gg \xi(t). \end{cases} \tag{8}$$

Assuming a power-law divergence for the persistence correlation length with t, i.e. $\xi(t) \sim t^Z$, then leads to the scaling relation $Z\alpha = \theta$. As we are working with the pure 2D Ising model at zero temperature, we expect [11] $Z = 1/2$; our results are completely consistent with this assumption.

To examine the correlated region ($r \ll \xi(t)$) we study the average number of persistent spins, $n(l,t)$, in square grids with dimensions $l \times l$ containing *at least* one such persistent spin. As

$$n(l,t) \approx \int_0^l C(r,t) r \, \mathrm{d}r \tag{9}$$

we have that

$$n(l,t) \sim \begin{cases} l^{2-\alpha} & \text{for} \quad l \ll \xi(t) \\ l^2 P(t) & \text{for} \quad l \gg \xi(t). \end{cases} \tag{10}$$

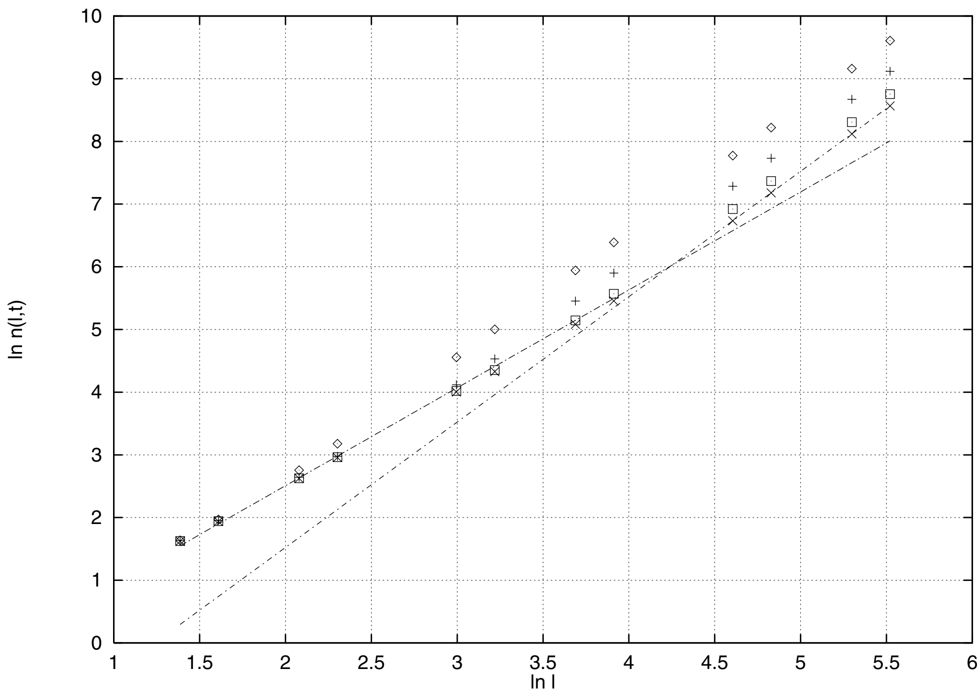

Figure 3. A log–log plot of $n(l, t)$ against l. Here, $n(l, t)$ is the average number of persistent spins in a square ($l \times l$) grid containing at least one persistent spin at time t. The data are shown for (top) $t = 10^2$ ◇, 10^3 +, 5×10^3 □ and 10^4 × (bottom). There is a clear crossover at $l \approx \sqrt{t}$ from a fractal distribution with dimension $d_f \sim 1.58$ to a homogeneous one with $d_f = d = 2$. The two straight lines (with slopes 1.58 and 2.00) are fits of the data in the two extreme cases for $t = 10^4$.

Hence, we expect the persistent spins to form a fractal with dimension $d_f = 2 - \alpha$ for length scales $l \ll \xi(t)$; the distribution is homogeneous on longer length scales, namely for $l \gg \xi(t)$. We expect the crossover to occur at $l \approx \xi(t) \sim t^{1/2}$. The scaling form for $n(l, t)$ is given by

$$n(l, t) = l^2 P(t) g(l/\xi(t)) \tag{11}$$

where $g(x)$ is a scaling function satisfying

$$g(x) \sim \begin{cases} x^{-\alpha} & \text{for} \quad x \ll 1 \\ 1 & \text{for} \quad x \gg 1. \end{cases} \tag{12}$$

We now discuss our results.

Figure 1 shows a plot of the scaling function $f(x)$ ($=C(r, t)/P(t)$) against $x = r/\xi(t)$ for various different values of t. We have assumed that $\xi(t) \sim t^{1/2}$. The data in figure 1 range over almost three orders of magnitude and are clearly consistent with this assumption. The large-x behaviour of $f(x)$ clearly follows the expected behaviour given in equation (7).

To extract a value for α we re-plot the data shown in figure 1 on a log–log scale in figure 2. The algebraic behaviour for $x \ll 1$ of the scaling function is confirmed by the linear fit. The slope of the straight line implies a value of $\alpha = 0.428 \pm 0.007$. Hence, the scaling relation would suggest that $\theta = Z\alpha = 0.214 \pm 0.004$. This is, of course, consistent with the value (0.209 ± 0.002) quoted above for θ [5].

We investigate the correlated regions by obtaining a direct estimate of the fractal dimension d_f. This is undertaken by first partitioning the lattice into square grids of size $l \times l$ with l ranging

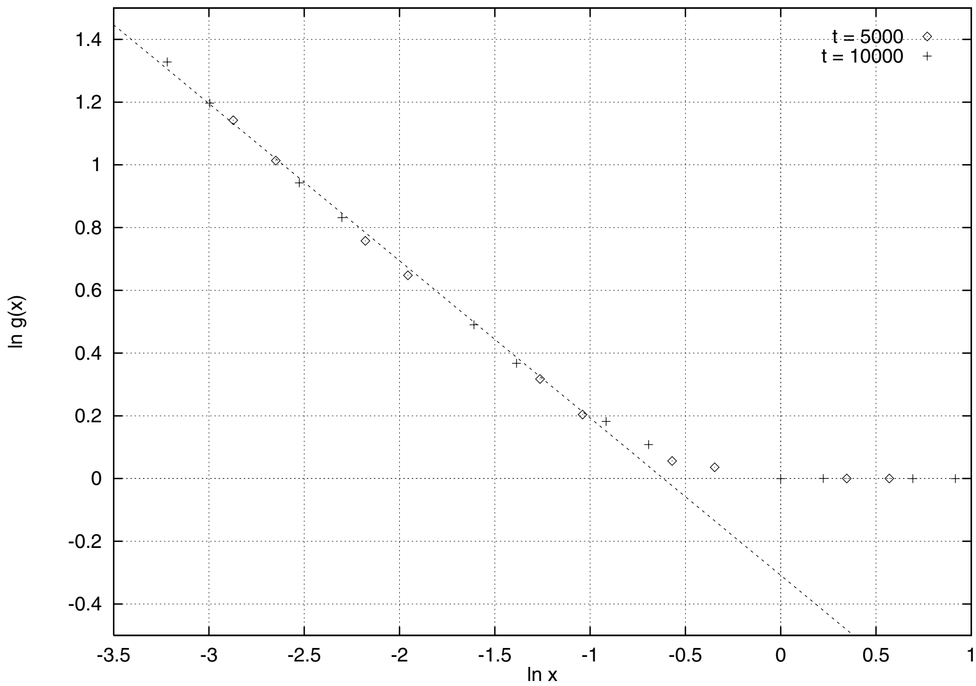

Figure 4. A plot of $\ln g(x)$ against $\ln x = \ln(l/\xi(t))$. Here the scaling function $g(x) = n(l,t)/l^2P(t)$. The straight line has slope $= -0.50$ and implies a value of $\alpha \sim 0.50$. Fitting the data over the *same* range as in figure 3 leads to $\alpha \sim 0.438$.

from 4 to 250. The average number of persistent spins in each $l \times l$ square containing at least one persistent spin is then obtained.

In figure 3 we plot $\ln n(l,t)$ versus $\ln l$ for $t = 10^2$, 10^3, 5×10^3 and 10^4. We notice that, for each of the values of t, the behaviour over sufficiently small (typically, $l \ll \sqrt{t}$) length scales is consistent with a fractal dimension $d_f = 2 - \alpha \sim 1.58$; over longer length scales (typically, $l \gg \sqrt{t}$) we retrieve homogeneous behaviour ($d_f = d = 2$). Actual values of d_f range from $d_f(t = 10^2) \sim 1.62$ to $d_f(t = 10^4) \sim 1.58$. The straight lines, with slopes 1.58 and 2.00, shown in figure 3 are linear fits to the behaviour in the two respective regimes for $t = 10^4$.

We obtain an independent estimate for the exponent α by re-plotting the data for $t = 5 \times 10^3$ and 10^4 in scaling form. Figure 4 shows a log–log plot of the scaling function $g(x) = n(l,t)/l^2P(t)$ against x where $x = l/\sqrt{t}$. We see that the data clearly fall onto a single scaling curve consistent with the expected behaviour given in equation (12). On fitting all of the data for $\ln x < -0.5$ we obtain a value of $\alpha \sim 0.438$. However, restricting the linear fit to $\ln x < -1$, as indicated by the straight line in figure 4, would imply a value of $\alpha \sim 0.50$. This value of α is slightly higher than the one we obtained from the analysis of the scaling behaviour of the two-point correlator (see equation (8)) and is a direct consequence of the restricted fit. It is emphasized that fitting the data over the *same* range as in figure 3 results in the lower value ($\alpha \sim 0.438$), which is clearly consistent with our result for the fractal dimension in the correlated regime.

To conclude, we have investigated the spatial distribution of persistent spins at zero temperature in the pure two-dimensional Ising model. We find that the persistent spins form a fractal with dimension $d_f \sim 1.58$ for length scales $r \ll \xi(t)$, where $\xi(t) \sim t^Z$ is the persistence correlation length. Furthermore, the persistence exponent satisfies the scaling relation $\theta = Z(2 - d_f)$ with $Z = 1/2$.

Acknowledgments

The simulations were performed partly on the SGI Origin 2000 at the University of Manchester made available by the Engineering and Physical Sciences Research Council (EPSRC), Great Britain, and also on in-house workstations and a PC. HF would like to thank the University of Derby for a research studentship.

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